Ruled fields and elementary equivalence

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Elementary equivalence versus Isomorphism

Question (Pop 2002)

1. (Arithmetic variant)
   Let $K$ and $L$ be finitely generated fields (over prime fields). Suppose $K \equiv L$. Then $K \cong L$?

2. (Geometric variant)
   Let $K$ and $L$ be function fields over an algebraically closed field $\kappa$. Suppose $K \equiv L$. Then $K \cong L$?

In this talk we confine ourselves to arithmetic case.
Let $K$ be finitely generated fields (over a prime field). We define the constant field $\kappa$ of $K$ to be algebraic closure of its prime field in $K$.

By function fields over $\kappa$, we mean finitely generated fields over $\kappa$ of transcendence degree $> 0$ over $\kappa$.

Every function fields $K/\kappa$ is a finite extension of $\kappa(t_1, \ldots, t_n)$, where $\{t_1, \ldots, t_n\}$ is a transcendence basis over $\kappa$.

Function fields arise naturally as the function fields of varieties over $\kappa$. For varieties $V$ over $\kappa$ we write $\kappa(V)$ for its function field.
1. Scanlon (2008) announced the affirmative answer to arithmetic variant, which turned out to be faulty (2011).

2. Thus these are still open, but some cases were proved affirmatively.
Former results of arithmetic variant

Theorem (Pop 2002, 2009?)

1. Let $K$ and $L$ be finitely generated fields (over prime fields) which are elementary equivalent. Suppose $K$ is of general type. Then $K \cong L$.

2. Let $K$ and $L$ be finitely generated fields (over prime fields) which are elementary equivalent. Suppose $K$ is a function field of a curve over a number field $\kappa$ (that is, $\text{trdeg}(K/\kappa) = 1$). Then $K \cong L$. 
Let $\kappa$ be perfect. Then every function fields $K/\kappa$ is a simple algebraic extension of $\kappa(t_1, \ldots, t_n)$, where $\{t_1, \ldots, t_n\}$ is a transcendence basis over $\kappa$. Thus $K$ is the function field of some hypersurface.

We say that $K/\kappa$ is of general type if it is the function field of a projective smooth variety $V$ over $\kappa$ of general type. $V$ is said to be of general type if the dimension equals to the Kodaira dimension. We have the following fact.

* A smooth hypersurface of dimension $n$ with degree $d$ is of general type iff $n > d + 2$

* If $K/\kappa$ is of general type and $f$ is a $\kappa$-endmorphism, $F$ is a $\kappa$-isomorphism.

Roughly speaking, almost all varieties are of general type.
We prove

**Theorem**

Let $\kappa$ be a finite fields, number fields. Let $K$ be a function field over prime fields. Suppose $K \equiv \kappa(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are indeterminates. Then $K \cong \kappa(x_1, \ldots, x_n)$.

$\kappa(x_1, \ldots, x_n)$ is a pure transcendental extension of $\kappa$ which is not of general type, since it is the function field of the hypersurface $Z = 0$ in $n + 1$-space which is of dimension $n$ and of degree 1.
We prove

**Theorem**

Let $\kappa$ be a finite field or a number field. Let $K$ be a separably uniruled field over $\kappa$ of $\text{trdeg}(K/\kappa) = 1$. Suppose $K \equiv L$. Then $K \cong L$. 
Cancellation problem for elementary equivalence

We prove

**Theorem**

Let $\kappa$ be a number field. Let $K, L$ be overfields of $\kappa$ with $\text{trdeg}(K/\kappa) = \text{trdeg}(L/\kappa) = 1$ such that $K(t) \equiv L(s)$ with $t$ transcendental over $K$ and $s$ transcendental over $L$. Suppose that $K/\kappa$ be not of genus 1. Then $K \cong L$. 
1. Let $K$ and $L$ be finitely generated fields (over prime fields) which are elementary equivalent. Let $K = k(t_1, \ldots, t_n, \alpha)$, where $k$ is the constant field of $K$, $\{t_1, \ldots, t_n\}$ is a transcendence basis over $k$ and $\alpha$ is algebraic over $k(t_1, \ldots, t_n)$. Let $f(t_1, \ldots, t_n, x)$ be the minimal polynomial over $k(t_1, \ldots, t_n)$. 
There is a uniform definition of "\( a_1, \ldots, a_l \) being algebraically independent over \( k \)" for each \( a_1, \ldots, a_l \) among finitely generated fields (over prime fields).
Hence constant fields and transcendency of tuples are definable.

Since the coefficients of \( f \) are in \( k \) and are definable, we have an embedding of \( K \) into \( L \). Likewise we have an embedding of \( K \) into \( L \).
Then we have an endomorphism \( f \) of \( K \). Since \( \text{Auto}(k) \) is finite, \( f^m \) is a \( k \)-endomorphism of \( K \) for some \( m \).
Since \( K \) is of general type, \( f^m \) is a \( k \)-isomorphism, \( f \) is an isomorphism hence \( K \cong L \).
Note that we only use $K \equiv_1 L$, that is, for every existential sentence $\varphi$, $K \models \varphi$ iff $L \models \varphi$. 
2. Pop gave a recipe which describes uniformly the $k$–valuations of function fields $K/k$ in one variable over number fields $k$. This allows us to give sentences $\varphi_K$ in the language of rings which describe the isomorphy type of $K$ among finitely generated fields.
Theorem

Let $\kappa$ be a finite fields, number fields. Let $K$ be a function field over prime fields. Suppose $K \equiv \kappa(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are indeterminates. Then $K \cong \kappa(x_1, \ldots, x_n)$. 

Proof

Let \( K = \kappa'(t_1, \ldots, t_n, u) \), where \( t_1, \ldots, t_n \) are a transcendental basis of \( K \) over \( \kappa' \) and \( \kappa \cong \kappa' \). Let \( F(\bar{t}, u) \) be the defining equation of \( K \). There is an embedding \( \iota \) of \( K \) into \( \kappa(x_1, \ldots, x_n) \). Hence there is a parametrization \( \phi_1(\bar{X}), \ldots, \phi_{n+1}(\bar{X}) \in \kappa'(\bar{X}) \) of \( F(\bar{X}, Y) \), where \( \bar{X} = X_1, \ldots, X_n \).

Suppose that \( K \) were not rational over \( \kappa' \), then for all \( \bar{a} \in K \), \( F(\phi_1(\bar{a}), \ldots, \phi_{n+1}(\bar{a})) \neq 0 \), which is first order definable.

Thus this sentence would hold in \( \kappa(x_1, \ldots, x_n) \), a contradiction.
Uniruled fields

Definition

Let $K/k$ be a field extension.

1. $K$ is called **ruled** over $k$ if there is a subfield $\Delta$ of $K$ containing $k$ so that $K = \Delta(t)$ for some element $t \in K$, where $t$ is transcendental over $\Delta$.

2. $K$ is called **uniruled** over $k$ if there is a finite extension field $L$ of $K$ so that $L$ is ruled over $k$. 
A fact on uniruled fields

We use the following fact.

Fact
Let $K/k$ be an extension of $\text{trdeg}(K/k)=1$ with $k$ algebraically closed in $K$. Then $K/k$ is separably uniruled iff there exists $x, y \in K$ and $a, b \in k$ such that $K = k(x, y)$ and

$\begin{align*}
x^2 - ay^2 &= b \quad \text{if } \text{char}(k) \neq 2 \\
x^2 + xy - ay^2 &= b \quad \text{if } \text{char}(k) = 2
\end{align*}$

Furthermore there is an element $c$ which is separably algebraic over $k$ and an element $t$ transcendental over $k$ such that $K(c) = k(c, t)$. 
Theorem

Let $\kappa$ be a finite field or a number field. Let $K$ be a separably uniruled field over $\kappa$ of $\text{trdeg}(K/\kappa) = 1$. Suppose $K \equiv L$. Then $K \simeq L$. 
Proof

There is an element $c$ which is separably algebraic over $\kappa$ and an element $t$ transcendental over $\kappa$ such that $K(c) = \kappa(c, t)$. Since $c$ is algebraic over $\kappa$, we have $K(c) \equiv L(c)$, where $K(c)$ is a rational field over $\kappa(c)$ which is a number field or a finite field. Hence we have $K(c) \cong L(c)$. Since $K$ and $\kappa(c)$ are linearly disjoint over $\kappa$ and so are $L$ and $\kappa(c)$, we have $K \cong L$. 
A fact on non-uniruled fields

We use the following fact.

Fact, Nagata

Let $K, L$ be overfields of $k$ and $K(t) \supseteq L$ with $t$ transcendental over $K$. Suppose that $L$ is not uniruled over $k$. Then $K \supseteq L$. 
We use the following fact.

Fact, Deveney

Let $K, L$ be overfields of $k$ with $\text{trdeg}(K/k) = \text{trdeg}(L/k) = 1$. Let $t$ be transcendental over $K$ and $s$ transcendental over $L$. Suppose that $K(t) = L(s)$. Then $K \cong L$. 
Fact

Let $K$ be a function field with $\text{trdeg}(K/k) = 1$. $K$ is of general type iff the genus of $K$ is more than 1. $K$ is uniruled iff its genus is 0.
Theorem

Let $\kappa$ be a number field. Let $K, L$ be overfields of $\kappa$ with $\text{trdeg}(K/\kappa) = \text{trdeg}(L/\kappa) = 1$ such that $K(t) \equiv L(s)$ with $t$ transcendental over $K$ and $s$ transcendental over $L$. Suppose that $K/\kappa$ be not of genus 1. Then $K \cong L$. 
Proof

First suppose that the genus of $K$ is more than 1. Since the coefficients of $f$ are in $k$ and are definable, we have an embedding of $K$ into $L(s)$. By Nagta’s theorem, we have an embedding of $K$ into $L$. Since $\text{trdeg}(K/\kappa) = \text{trdeg}(L/\kappa) = 1$ and $K$ is not uniruled, $L$ is also not uniruled.

On the other hand, we have an embedding of $L$ into $K(t)$. Similarly, we have an embedding of $L$ into $K$.

Since $K$ is of general type, we have $K \cong L$. 

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Next suppose that the genus of $K$ equals to 1, that is, uniruled over $\kappa$. Since $\kappa$ is a number field, $K/\kappa$ is separably uniruled. Hence there is an element $c$ which is separably algebraic over $\kappa$ and an element $t$ transcendental over $\kappa$ such that $K(c) = \kappa(c, t)$. Then $K(t, c)$ is rational over $\kappa(c)$. As before we have $K(c, t) \cong L(c, s)$ and we know that $L(c, s)$ is rational over $\kappa(c)$, that is, $K(c, t) \cong L(c, s)$. Let the image of $L(c, s)$ in $K(c, t)$ be $L'(c, s')$. Then $K(c, t) = L'(c, s')$. By Deveney’s theorem, we have $K(c) \cong L'(c)$ and $K \cong L'$ as before, hence $K \cong L$. 


6. Florian Pop, "Elementary equivalence versus isomorphism $1\frac{1}{2}$", (unpublished.)